

4.1

$X(t)$  and  $Y(t)$  independent and WSS with ACFs:

$$\left. \begin{aligned} r_x(\tau) &= \sigma_x^2 e^{-a|\tau|} + m_x^2 \\ r_y(\tau) &= \sigma_y^2 e^{-b|\tau|} + m_y^2 \end{aligned} \right\} \text{We define } Z(t) = X(t)Y(t)$$

$$\begin{aligned} \bullet \quad r_z(t, \tau) &= E[X(t+\tau)Y(t+\tau)X(t)Y(t)] = \left. \begin{aligned} &\text{independence} \\ &X(t) \text{ and } Y(t) \end{aligned} \right\} = \\ &= E[X(t+\tau)X(t)] E[Y(t+\tau)Y(t)] = \left. \begin{aligned} &Y(t) \text{ and } X(t) \text{ are} \\ &\text{WSS} \end{aligned} \right\} = \\ &= r_x(\tau) r_y(\tau). \end{aligned}$$

$$\bullet \quad R_z(f) = \mathcal{F}\{r_z(\tau)\} = R_x(f) * R_y(f).$$

therefore

$$r_z(\tau) = (\sigma_x^2 e^{-a|\tau|} + m_x^2)(\sigma_y^2 e^{-b|\tau|} + m_y^2)$$

We don't want to calculate the convolution of the two Fourier transforms. Hence, we will express  $r_z(\tau)$  in a way that we can easily compute its Fourier transform.

$$r_z(\tau) = \sigma_x^2 \sigma_y^2 e^{-(a+b)|\tau|} + m_x^2 m_y^2 + m_x^2 \sigma_y^2 e^{-b|\tau|} + m_y^2 \sigma_x^2 e^{-a|\tau|}$$

$$\Rightarrow \text{The Fourier transform of } e^{-a|\tau|} \rightarrow \frac{2a}{a^2 + (2\pi f)^2}$$

$$\Rightarrow \text{The Fourier transform of a constant } A \rightarrow A \delta(f)$$

$$R_z(f) = \sigma_x^2 \sigma_y^2 \frac{2(a+b)}{(a+b)^2 + (2\pi f)^2} + m_x^2 m_y \delta(f) + m_x^2 \sigma_y^2 \frac{2b}{b^2 + (2\pi f)^2} +$$

$$+ m_y^2 \sigma_x^2 \frac{2a}{a^2 + (2\pi f)^2}$$



4.81

AM signal  $S(t) = C_0 A(t) \cos(2\pi f_c t + \Phi)$

$\Phi$  R.v. uniform  $[0, 2\pi]$   
 $A(t)$ . stationary process  $r_A(\tau), R_A(f)$  } independent

a) Show  $S(t)$  is WSS

b) Obtain the mean of  $S(t)$

$$\begin{aligned} \bullet E[S(t)] &= E[C_0 A(t) \cos(2\pi f_c t + \Phi)] = C_0 E[A(t) \cos(2\pi f_c t + \Phi)] = \\ &= \left\{ \begin{aligned} E[\chi g(X)] &= \iint g(x) \cdot \chi f_{Xg}(x, y) dx dy = \{ \text{independence} \} = \\ &= \iint g(x) \chi f_X(x) f_Y(y) dx dy = \int g(x) f_X(x) dx \int \chi f_Y(y) dy = \\ &= E[\chi] E[g(X)] \end{aligned} \right\} = \end{aligned}$$

$$\begin{aligned} &= C_0 E[A(t)] E[\cos(2\pi f_c t + \Phi)] = m_A \cdot \int_0^{2\pi} \cos(2\pi f_c t + \Phi) \cdot \frac{1}{2\pi} d\Phi = \\ &= \frac{m_A}{2\pi} \sin(2\pi f_c t + \Phi) \Big|_0^{2\pi} = \frac{m_A}{2\pi} (\sin(2\pi f_c t + 2\pi) - \sin(2\pi f_c t)) = 0 \end{aligned}$$

$$\begin{aligned} \bullet E[S(t+\tau) S(t)] &= C_0^2 E[A(t+\tau) \cos(2\pi f_c (t+\tau) + \Phi) A(t) \cos(2\pi f_c t)] = \\ &= \{ \text{independence } A(t) \text{ and } \Phi \} = C_0^2 E[A(t+\tau) A(t)] E[\cos(2\pi f_c (t+\tau) + \Phi) \cdot \\ &\cdot \cos(2\pi f_c t)] = C_0^2 r_A(\tau) E\left[\frac{1}{2} \cos(2\pi f_c (2t+\tau) + 2\Phi) + \frac{1}{2} \cos(2\pi f_c \tau)\right] = \\ &= r_A(\tau) \left( \int_0^{2\pi} \frac{1}{2} \cos(2\pi f_c (2t+\tau) + 2\Phi) \frac{1}{2\pi} d\Phi + \frac{1}{2} \cos(2\pi f_c \tau) \right) C_0^2 = \\ &= C_0^2 r_A(\tau) \left( \frac{1}{8\pi} \sin(2\pi f_c (2t+\tau) + 2\Phi) \Big|_0^{2\pi} + \frac{1}{2} \cos(2\pi f_c \tau) \right) = \\ &= C_0^2 r_A(\tau) \left( 0 + \frac{1}{2} \cos(2\pi f_c \tau) \right) \quad \left/ \quad S(t) \text{ is stationary} \right. \end{aligned}$$

c)  $R_S(f)$  of  $S(t)$ .

$$r_S(\tau) = r_A(\tau) \left( \frac{1}{2} \cos(2\pi f_c \tau) \right) C_0^2$$

$$R_S(f) = R_A(f) * \left( \frac{1}{4} \delta(f - f_c) + \frac{1}{4} \delta(f + f_c) \right) C_0^2$$

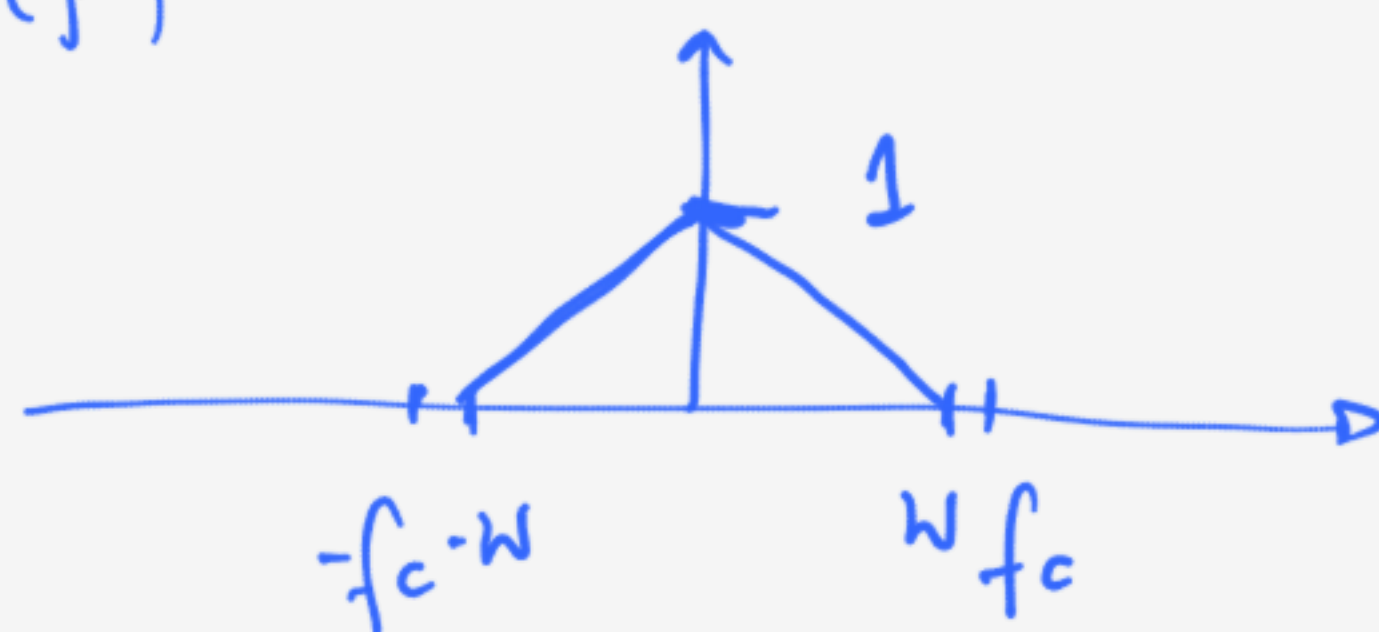
$$R_S(f) = \frac{1}{4} (R_A(f - f_c) + R_A(f + f_c)) C_0^2$$



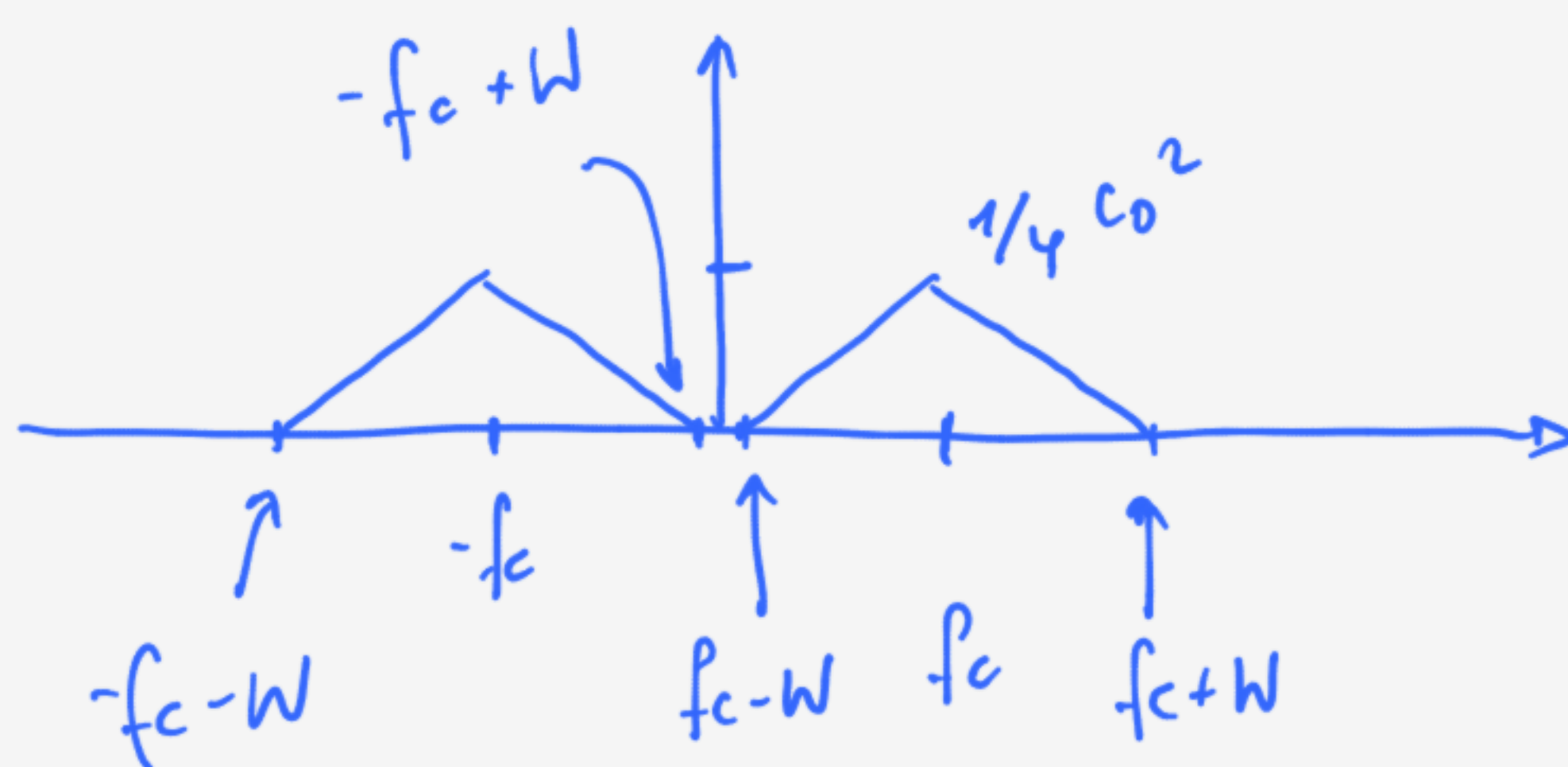
$$R_S(f) = \frac{1}{4} (R_A(f+f_c) + R_A(f-f_c)) c_0^2$$

$$R_A(f) = 0 \text{ when } |f| > W, \quad W < f_c$$

Example of  $R_A(f)$



Then  $R_S(f)$



4.9] Determine  $R(v)$  of

$$b) \quad r(k) = f(k) + \frac{1}{|k|!}$$

$$R(v) = \sum_{k=-\infty}^{\infty} (f(k) + \frac{1}{|k|!}) e^{-j2\pi kv} = \sum_{k=-\infty}^{-1} \frac{1}{|k|!} e^{-j2\pi kv} + (f(0) + \frac{1}{0!}) e^0$$

$$+ \sum_{k=1}^{\infty} \frac{1}{|k|!} e^{j2\pi kv} = \left\{ \sum_{k=-\infty}^{-1} \frac{1}{|k|!} e^{-j2\pi kv} = \sum_{k=1}^{\infty} \frac{1}{k!} e^{j2\pi kv} \right\} \quad \begin{matrix} \text{0! is 1} \\ \text{convention} \end{matrix}$$

$$= \sum_{k=1}^{\infty} \frac{1}{k!} (e^{j2\pi kv} + e^{-j2\pi kv}) + 2 =$$

$$= \sum_{k=1}^{\infty} \frac{1}{k!} ((e^{j2\pi v})^k + (e^{-j2\pi v})^k) + 2 =$$

$$= \left\{ \text{Maclaurin's series } e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \right. \quad \left. \text{(Taylor extension for}$$

complex space)

$$= e^{e^{j2\pi v}} + e^{e^{-j2\pi v}} = e^{\cos(2\pi v) + j\sin(2\pi v)} + e^{\cos(2\pi v) - j\sin(2\pi v)} =$$



$$\begin{aligned}
&= e^{\cos(2\pi\nu)} e^{+j\sin(2\pi\nu)} + e^{\cos(2\pi\nu)} e^{-j\sin(2\pi\nu)} = \\
&= e^{\cos(2\pi\nu)} (e^{j\sin(2\pi\nu)} + e^{-j\sin(2\pi\nu)}) = \\
&= e^{\cos(2\pi\nu)} (2\cos(\sin(2\pi\nu))) \quad \text{J}
\end{aligned}$$

4.10] ACF of process with spectral density:

$$b) R(\nu) = \frac{9}{10 - 6\cos(2\pi\nu)} = \left\{ \text{DFT}(a^{|n|})(\nu) = \frac{1-a^2}{1+a^2-2a\cos(2\pi\nu)} \right\}$$

We need to find  $a$  and  $b$  so that:

$$\frac{9}{10 - 6\cos(2\pi\nu)} = \frac{c(1-a^2)}{b(1+a^2-2a\cos(2\pi\nu))}$$

For the denominator to be equal:

$$9 = c(1-a^2) \rightarrow$$

$$10 = b(1+a^2)$$

$$-6\cos(2\pi\nu) = -2ab\cos(2\pi\nu) \Rightarrow -6 = -2ab$$

$$3 = ab \quad b = 3/a$$

$$10 = b(1+a^2) = \frac{3}{a}(1+a^2)$$

$$10a = 3(1+a^2) \rightarrow 3a^2 - 10a + 3 = 0$$

$$a = \frac{10 \pm \sqrt{(100) - 36}}{6} = \frac{10 \pm 8}{6} \quad \begin{matrix} \nearrow \frac{2}{6} = \frac{1}{3} \text{ J} \\ \searrow \frac{18}{6} = 3 \text{ J} \end{matrix}$$

two possible values for  $a$  will lead to other two possible values of  $b$  and  $c$  (one for each value of  $a$ ).

$\Rightarrow$  Note that we want the function to be an acf. function!

If we select  $a=3$  the function is increasing with  $k$  which implies it cannot be an ACF! Hence, we must choose  $a=1/3$ .



If  $a = 1/3$  we have that  $3 = ab \Rightarrow b = 9$

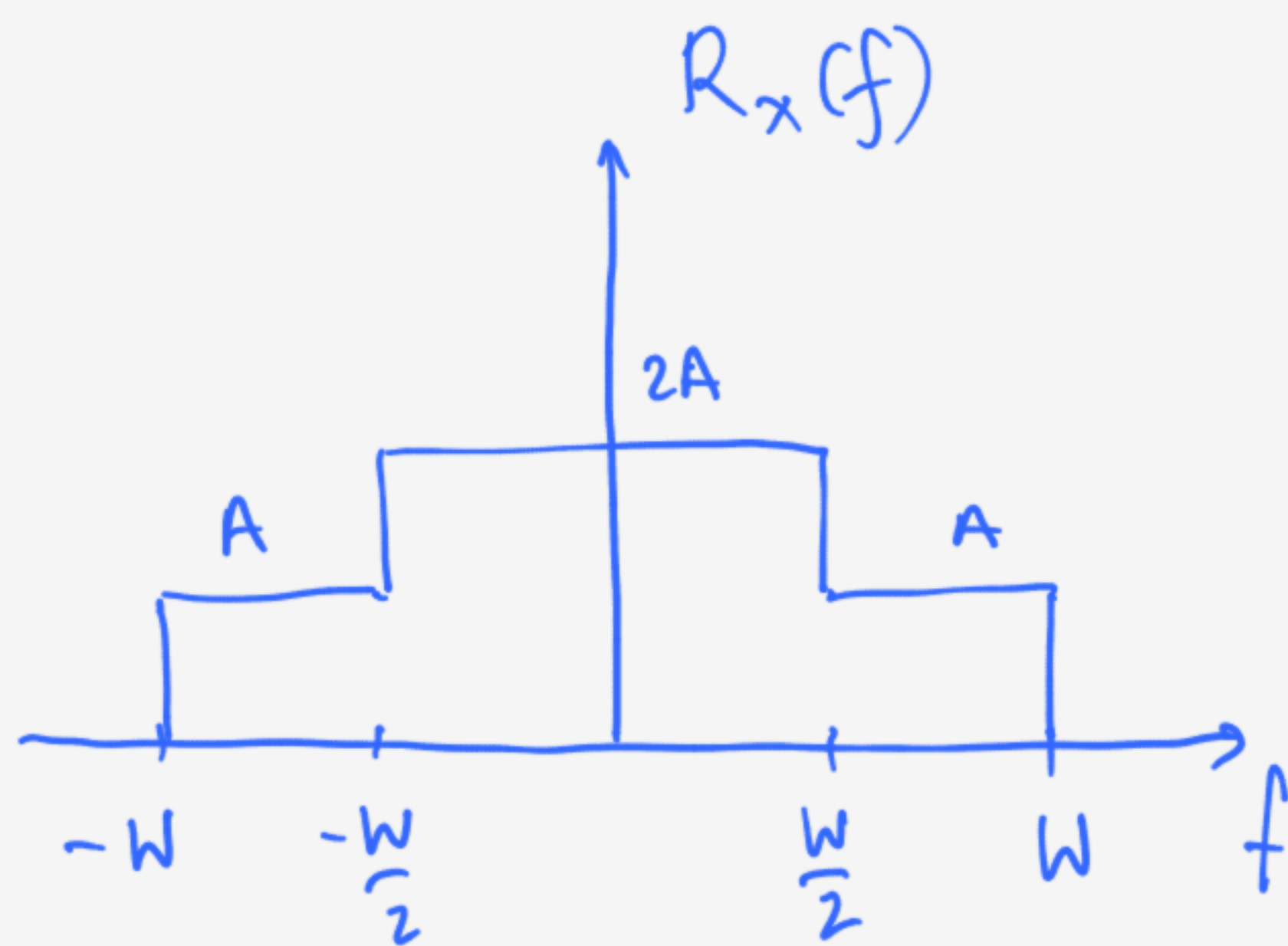
and  $9 = c(1 - a^2) \Rightarrow 9 = c(1 - \frac{1}{9})$

$$9 = c \left( \frac{8}{9} \right) \quad c = \frac{81}{8}$$

Hence we have that the DFT<sup>-1</sup> of  $R(p)$  is

$$r_p(n) = \frac{c}{b} \left( \frac{1}{3} \right)^{|n|} = \frac{81/8}{9} (3^{-1})^{|n|} = \frac{9}{8} 3^{-|n|}$$

4.11]  $x(t)$  has the power spectral density:



Determine its ACF  $r_x(z)$

$$R_x(f) = A \Pi\left(\frac{f}{W}\right) + A \Pi\left(\frac{f}{2W}\right)$$

Then we have that:

$$x(ct) \ (c \neq 0) \xrightarrow{\mathcal{F}} \frac{1}{|c|} x\left(\frac{f}{c}\right)$$

$$\text{sinc}(t) \xrightarrow{\mathcal{F}} \Pi(f)$$

$$r_x(\tau) = AW \text{sinc}(W\tau) + A2W \text{sinc}(2W\tau)$$